Transformations $RS_4^2(3)$ of the Ranks ≤ 4 and Algebraic Solutions of the Sixth Painlevé Equation

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July 31, 2001

Abstract

Compositions of rational transformations of independent variables of linear matrix ordinary differential equations (ODEs) with the Schlesinger transformations (RS-transformations) are used to construct algebraic solutions of the sixth Painlevé equation. RS-Transformations of the ranks 3 and 4 of 2×2 matrix Fuchsian ODEs with 3 singular points into analogous ODE with 4 singular points are classified.

2000 Mathematics Subject Classification: 34M55, 33E17, 33E30.

Short title: RS-transformations

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1 Introduction

A considerable attention has been paid to the study of algebraic solutions of the sixth Painlevé equation [1, 2, 3, 4, 5]. Recently, in this connection, a general method for construction of the so-called special functions of the isomonodromy type (SFITs)[6], which algebraically depend on one of their variables, was suggested in [7]. SFITs include, in particular, functions of the hypergeometric and Painlevé type. With each SFIT there is an associated matrix linear ODE such that the SFIT defines isomonodromy deformation of this linear ODE. A key point of the proposed method is a construction of the so-called $RS^n_m(k)$ - transformations. These transformations map fundamental solution of $n \times n$ matrix linear ODEs with k-singular points into analogous ODEs with m singular points. Each RS-transformation is a composition of a rational transformation of the argument of a given linear ODE with an appropriate Schlesinger transformation. These transformations preserve isomonodromic property, therefore they generate transformations of the corresponding SFITs. An important parameter of the RS-transformations is their rank, r, which, by definition, equals to the order of the corresponding rational transformation, i.e., the number of its preimages counted with their multiplicities.

In this paper we classify $RS^2_4(3)$ transformations of the ranks 3 and 4 for the Fuchsian ODEs. The number of intrinsic parameters of these transformations define whether they generate: (1) explicit mappings of very simple SFITs, namely constants, to the solutions of the sixth Painlevé equation (P_6); or (2) special sets of numbers which can be interpreted as initial data of particular solutions to P_6 given at some special points of the complex plane. In fact, the solutions of the first type are nothing but algebraic solutions to P_6, whilst the second ones are, in general, transcendental solutions whose initial data at some particular points of the complex plane can be calculated explicitly in terms of the monodromy data of the associated linear matrix Fuchsian ODE. Although solutions of the second type are, in general, transcendental (non-classical in the sense of Umemura [8]), they are "less transcendental" than the other non-classical solutions of P_6: due to the above mentioned relation of the initial and monodromy data one can obtain asymptotic expansions of these solutions in the neighborhood of the singular points of P_6, 0, 1, and ∞, in terms of their initial data provided the latter are given at the special points. Actually, an example of solutions of the second type was recently discussed by Manin [9] in connection with some special construction of the Frobenius manifold. The reader can find some further details about this solution in Subsection 3.2.1.

Consider the following Fuchsian ODE with three singular points, 0, 1, and ∞:

$$\frac{d\Phi}{d\mu} = \left(\frac{\tilde{A}_0}{\mu} + \frac{\tilde{A}_1}{\mu - 1} \right) \Phi, \tag{1.1}$$

where $\tilde{A}_0$ and $\tilde{A}_1$ are 2×2 matrices independent of μ. Let us denote $\det(A_p) = -\frac{\eta_p^2}{4}$, $p = 0, 1$. We suppose that Eq. (1.1) is normalized as follows: $\mathrm{tr}\tilde{A}_p = 0$, $\tilde{A}_0 + \tilde{A}_1 = -\frac{\eta_\infty}{2}\sigma_3$ where σ_3 is the Pauli matrix: $\mathrm{diag}\{1, -1\}$. Given parameters η_q, $q = 0, 1, \infty$ matrices $\tilde{A}_q$ are fixed up to a diagonal gauge, $c^{\sigma_3} A_p c^{-\sigma_3}$, $c \in \mathbb{C} \setminus 0$. It is well known, that solutions of Eq. (1.1) can be written explicitly in terms of the Gauß hypergeometric function (see, e.g. [12]), however, we don't use this representation here. In this paper we consider RS-transformations of Eq. (1.1) into the following 2×2 matrix ODE with four Fuchsian

singular points,

$$\frac{d\Psi}{d\lambda} = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + \frac{A_t}{\lambda-t}\right)\Psi, \tag{1.2}$$

where matrices A_l, $l = 0, 1, t$ are independent of λ. Moreover, we assume the following normalization of Eq. (1.2), $\mathrm{tr} A_l = 0$, $A_0 + A_1 + A_t = -\frac{\theta_\infty}{2}\sigma_3$. We also denote $\det A_l = -\frac{\theta_l^2}{2}$. It is now well known, [11], that isomonodromy deformations of Eq. (1.2) in general situation (for details see [13]) define solutions of the sixth Painlevé equation, P_6,

$$\begin{aligned}\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \\ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha_6 + \beta_6\frac{t}{y^2} + \gamma_6\frac{t-1}{(y-1)^2} + \delta_6\frac{t(t-1)}{(y-t)^2}\right),\end{aligned} \tag{1.3}$$

where α_6, β_6, γ_6, $\delta_6 \in \mathbb{C}$ are parameters. We recall the relation between isomonodromy deformations of Eq. (1.2) and Eq. (1.3) following the work [11]. Denote by A_k^{ij} the corresponding matrix elements of A_k. Note, that due to the normalization

$$A_0^{12} + A_1^{12} + A_t^{12} = A_0^{21} + A_1^{21} + A_t^{21} = 0,$$

therefore equations,

$$\frac{A_0^{ik}}{y_{ik}} + \frac{A_1^{ik}}{y_{ik}-1} + \frac{A_t^{ik}}{y_{ik}-t} = 0,$$

for $\{ik\} = \{12\}$ and $\{ik\} = \{21\}$ have in general situation ($A_1^{ik} + tA_t^{ik} \neq 0$) unique solutions y_{ik}. These functions solve Eq. (1.3) with the following values of the parameters,

$$y_{12}(t): \quad \alpha_6 = \frac{(\theta_\infty - 1)^2}{2}, \;\; \beta_6 = -\frac{\theta_0^2}{2}, \;\; \gamma_6 = \frac{\theta_1^2}{2}, \;\; \delta_6 = \frac{1-\theta_t^2}{2}, \tag{1.4}$$

$$y_{21}(t): \quad \alpha_6 = \frac{(\theta_\infty + 1)^2}{2}, \;\; \beta_6 = -\frac{\theta_0^2}{2}, \;\; \gamma_6 = \frac{\theta_1^2}{2}, \;\; \delta_6 = \frac{1-\theta_t^2}{2}. \tag{1.5}$$

One can associate with isomonodromy deformations of Eq. (1.2) one more function, the so-called τ-function [14], which plays a very important role in applications. This function [11] is defined via the function σ,

$$\sigma(t) = \mathrm{tr}(((t-1)A_0 + tA_1)A_t) + t\kappa_1\kappa_2 - \frac{1}{2}(\kappa_3\kappa_4 + \kappa_1\kappa_2),$$

where

$$\kappa_1 = \frac{\theta_t + \theta_\infty}{2}, \;\; \kappa_2 = \frac{\theta_t - \theta_\infty}{2}, \;\; \kappa_3 = -\frac{\theta_1 + \theta_0}{2}, \;\; \kappa_4 = \frac{\theta_1 - \theta_0}{2}.$$

The function σ solves the following ODE,

$$t^2(t-1)^2\sigma''^2\sigma' + \left(2\sigma'(t\sigma' - \sigma) - \sigma'^2 - \kappa_1\kappa_2\kappa_3\kappa_4\right)^2 = (\sigma' + \kappa_1^2)(\sigma' + \kappa_2^2)(\sigma' + \kappa_3^2)(\sigma' + \kappa_4^2),$$

where the prime is differentiation with respect to t. The τ-function is defined (up to a multiplicative constant) as the general solution of the following ODE,

$$t(t-1)\frac{d}{dt}\ln\tau = \sigma(t).$$

We explain now a general idea of how to construct RS-transformations of Eq. (1.1) to Eq. (1.2). Consider a rational transformation of the argument,

$$\mu \equiv \mu(\lambda) = P(\lambda)/Q(\lambda), \tag{1.6}$$

where $P(\lambda)$ and $Q(\lambda)$ are mutually prime polynomials. The function $\mu(\lambda)$ maps Eq. (1.1) with $k = 3$ singular points, 0, 1, ∞, into an intermediate Fuchsian ODE (on λ) with $3r$ singular points, where r is the rank of the RS-transformation, $r = \max(\deg\{P\}, \deg\{Q\})$. If some of the parameters, η_p, $p = 0, 1, \infty$, are rational numbers, say, $\eta_0 = n_0/m_0$ with the mutually prime natural numbers n_0 and $m_0 \geq 2$, and mapping (1.6) is chosen such that some preimage of 0 has the multiplicity proportional to m_0, then this preimage is a Fuchsian singular point of the intermediate ODE with the monodromy matrix proportional to $\pm I$. Therefore, the latter singular point is removable by a proper Schlesinger transformation (an apparent singularity). Thus, the idea is to choose the parameters η_p of Eq. (1.1) and rational mapping (1.6) such that $3r - 4$ points of the intermediate equation can be removed so that one finally arrives to Eq. (1.2).

We classify RS-transformations up to fractional-linear transformations of μ permutating the points 0, 1, and ∞ and also up to fractional-linear transformations of the variable λ defining transformations of the set of singular points of Eq. (1.2), 0, 1, ∞, and t, into 0, 1, ∞, and $\tilde{t}$. Clearly, $\tilde{t}$ is one of the points of the orbit, t, $1/t$, $1-t$, $1-1/t$, $1/(1-t)$, $t/(1-t)$. In terms of the algebraic solutions of the sixth Painlevé equation fractional-linear transformations of μ are nothing but reparametrizations of these solutions, whilst fractional-linear transformations of λ generally define superpositions of so-called Bäcklund transformations of the solutions to Eq. (1.3) with corresponding fractional-linear transformations of y and t. As soon as some RS-transformation is constructed one can construct infinite number of such transformations which differ only by a finite number of Schlesinger transformations preserving singular points: 0, 1, t, and ∞, of Eq. (1.2). We call such RS-transformations equivalent and construct, in most cases, only one transformation representing the whole class.

For a classification of the RS-transformations of the rank r consider partitions of r. Multiplicities of preimages of 0, 1, ∞ of the rational mapping (1.6) is a triple of partitions of r. Möbious invariance in μ means that we don't distinguish triples which differ only by the ordering of the partitions. All in all there are $\frac{1}{6}N_r(N_r+1)(N_r+2)$ of such triples, where N_r is the total number of partitions of r (a number of the corresponding Young tableaux). Our method of classification of the RS-transformations can be regarded as a selection of the proper triples of the Young tableaux. It consists of three steps:

1. A sieve-like procedure with the goal to get rid of the triples which generate more than $m = 4$ non-removable singularities of the intermediate ODE. To calculate the number of triples which pass through the sieve let us introduce some notation. For each partition $\mathcal{P}$ denote $\mathcal{M}$ a maximal subset with the greatest common divider greater than 1. In the case when there are several such subsets, $\mathcal{M}$ is anyone of them. Denote a_j a number of the Young tableaux with $\mathrm{card}(\mathcal{P} \setminus \mathcal{M}) = j$. Clearly, the sieve is passed by those triples which satisfy the condition, $j_1 + j_2 + j_3 \leq 4$, where j_k means a value of the parameter j for the k-th partition of the triple. The total number of the triples which pass through the sieve is

$$\begin{aligned}&\tfrac{a_0(a_0+1)}{2}\left(\tfrac{a_0+2}{3} + a_1 + a_2 + a_3 + a_4\right) + \tfrac{a_1(a_1+1)}{2}\left(a_0 + \tfrac{a_1+2}{3} + a_2\right)\\ &+\tfrac{a_2(a_2+1)}{2}a_0 + a_0a_1(a_2+a_3) - 1;\end{aligned} \tag{1.7}$$

The last term in Eq. (1.7), -1, is related with the fact that equation

$$x^r + y^r = z^r$$

has no solutions in mutually prime polynomials, x, y, and z, for $r \geq 2$.

2. The aim of this stage is to choose among the triples selected at the first step those ones for which there exist corresponding rational mappings (1.6). Denote $i = \mathrm{card}\mathcal{M}$, then Eq. (1.2) with arbitrary parameter t can exist only in the case when
$$i_1 + j_1 + i_2 + j_2 + i_3 + j_3 \geq r + 3, \tag{1.8}$$
where the subscripts denote parameters i and j characterizing partitions in the triple. In the case
$$i_1 + j_1 + i_2 + j_2 + i_3 + j_3 = r + 2 \tag{1.9}$$
only RS-transformations with a parameter t equal to some special number could exist;

3. The final stage is the construction of the RS-transformation. This stage includes an analysis of how many RS-transformations can be constructed for a given partition: sometimes there are few different RS - transformations due to the ambiguity of the choice of the set $\mathcal{M}$ for some partitions.

One more question which sounds naturally in connection with the RS-transformations is as follows: which transformations of the rank r can be presented as a superposition of RS-transformations of lower ranks r_1, r_2,...r_N? Clearly in the latter case one writes
$$r = r_1 r_2 \cdot \ldots \cdot r_N.$$

The classification of RS-transformations of the rank 2, 3 and 4 is given in the following sections.

2 RS-transformations of the rank 2

We apply the scheme suggested in Introduction for $r = 2$ ($N_2 = 2$). After the first stage from the total number $4 = \frac{1}{6} \cdot 2 \cdot 3 \cdot 4$ of the triples of partitions of r we are left with 2 triples, $(2|1+1|1+1)$ and $(2|1+1|2)$, since $a_0 = a_2 = 1$ and $a_1 = a_3 = a_4 = 0$. At the second stage we construct rational mappings corresponding to these triples. The rational mapping which corresponds to the second triple clearly has a very simple form, $\mu = \lambda^2$. The parameter $t = -1$, so that this transformation does not generate any algebraic solution of the sixth Painlevé equation. The latter fact also follows from Eq. (1.9), since $i_1 = i_3 = 1$, $j_1 = j_3 = 0$, $i_2 = 0$, and $j_2 = 2$. However, as it is mentioned in Introduction (see also [7]), this transformation can be interpreted as some property of special solutions of the sixth Painlevé equation.

To the first triple, more precisely, to the triple $(1+1|2|1+1)$, there corresponds the following rational mapping,
$$\mu = \rho\frac{\lambda(\lambda-1)}{\lambda - t} \qquad \text{and} \qquad \mu - 1 = \rho\frac{(\lambda - a)^2}{\lambda - t}, \tag{2.1}$$
where
$$t = \frac{s^2(s-1)^2}{(s^2-2s-1)(s^2+1)}, \quad a = \frac{s(s-1)}{s^2+1}, \quad \rho = \frac{s^2+1}{s^2-2s-1},$$
with arbitrary $s \in \mathbb{C}$. This fact is consistent with Eq. (1.8), in this case $i_1 = i_3 = 0$, $j_1 = j_3 = 2$, $i_2 = 1$, and $j_2 = 0$. Rational mapping (2.1) generates RS-transformation

which we denote $RS_4^2(1+1|2|1+1)$. This transformation exists for arbitrary parameters η_0, $\eta_\infty \in \mathbb{C}$ and parameter $\eta_1 = 1/2$. and reads as follows,

$$\Psi(\lambda) = \left(J_\infty^* \sqrt{\lambda - a} + J_a^* \frac{1}{\sqrt{\lambda - a}} \right) \Phi(\mu),$$

where

$$J_\infty^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_a^* = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(2\eta_\infty+2\eta_0+1)(s^2-2s-1)}{16\eta_\infty(\eta_\infty+1)(s^2+1)} \end{pmatrix} \begin{pmatrix} 1 & \frac{2\eta_\infty-2\eta_0+1}{2\eta_\infty+2\eta_0-1} \\ 2\eta_\infty+2\eta_0-1 & 2\eta_\infty-2\eta_0+1 \end{pmatrix}.$$

The values of the θ-parameters are,

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_\infty, \quad \theta_\infty = \eta_\infty + 1.$$

Corresponding algebraic solutions of the sixth Painlevé equation and related functions $\sigma(t)$ and $\tau(t)$ are as follows:

$$y_{12}(t) = \tfrac{(s-1)s}{s^2-2s-1} = t + \sqrt{t^2 - t},$$
$$y_{21}(t) = y_{12}(t) \tfrac{((2\eta_\infty+3+2\eta_0)(s^2+1)-(2+4\eta_0)(s+1))((2\eta_\infty+3-2\eta_0)(s^2+1)-(2-4\eta_0)(s+1))}{((2\eta_\infty+3)^2-4\eta_0^2)(s^2+1)^2-4(4\eta_0^2-1)(s^3-s)},$$

$$\begin{aligned} \sigma(t) &= -\frac{(4\eta_0^2 + (1+2\eta_\infty)^2)(s^2+1)^2 - 16\eta_0^2(s^3-s)}{16(s^2+1)(s^2-2s-1)}, \\ \tau(t) &= (s^3-s)^{-\frac{1}{2}\eta_0^2 - \frac{1}{2}(\frac{1}{2}+\eta_\infty)^2}(s^2-2s-1)^{(\frac{1}{2}+\eta_\infty)^2}(s^2+1)^{\eta_0^2}. \end{aligned}$$

Remark 1 *Hereafter we omit the multiplicative parameter C in formulae for the τ-function ($\tau(t) \to C\tau(t)$).*

Since the function y_{12} is independent of the parameters η_0 and η_∞ corresponding terms in Eq. (1.3), proportional to η_0^2 and η_∞^2 should vanish for this solution. Therefore, y_{12} solves the following algebraic equation:

$$\frac{t}{y_{12}^2} - \frac{t-1}{(y_{12}-1)^2} = 0 \quad \text{or, equivalently,} \quad 1 - \frac{t(t-1)}{(y_{12}-t)^2} = 0.$$

3 RS-transformations of the rank 3

In the case $r = 3$ ($N_3 = 3$) the total number of different triples is $\frac{1}{6} \cdot 3 \cdot 4 \cdot 5 = 10$. According to Eq. (1.7) five triples survive after the first stage, since $a_0 = a_1 = a_3 = 1$ and $a_2 = a_4 = 0$. They are $(1+1+1|2+1|3)$, $(1+1+1|3|3)$, $(2+1|2+1|2+1)$, $(2+1|2+1|3)$, and $(2+1|3|3)$. As it follows from Eqs. (1.8) and (1.9) two triples, $(1+1+1|2+1|3)$ and $(2+1|2+1|2+1)$, define RS-transformations with the arbitrary parameter t; two triples, $(1+1+1|3|3)$ and $(2+1|2+1|3)$, define RS-transformations with fixed t; and the last triple, $(2+1|3|3)$, does not define any RS-transformation.

3.1 RS-transformations with arbitrary t

3.1.1 $RS_4^2(2+1|2+1|2+1)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda-t)^2}{(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \rho\frac{(\lambda-1)(\lambda-c)^2}{(\lambda-b)^2},$$

where

$$t = 1 - s^2, \quad c = (s+1)^2, \quad \rho = \frac{1}{(2s+1)^2}, \quad b = \frac{(s+1)^2}{2s+1},$$

with arbitrary $s \in \mathbb{C}$. We define RS-transformation for arbitrary value of $\eta_0 \in \mathbb{C}$, $\eta_1 = 1/2$, and $\eta_\infty = -1/2$, as follows:

$$\Psi(\lambda) = \left(J_c\sqrt{\frac{\lambda-c}{\lambda-b}} + J_b\sqrt{\frac{\lambda-b}{\lambda-c}}\right)\Phi(\mu).$$

In the previous formula,

$$J_c = \begin{pmatrix} 0 & \frac{\eta_0}{\eta_0-1} \\ 0 & 1 \end{pmatrix}, \quad J_b = J_c^*.$$

Hereafter, we use the following matrix operation *,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

The θ-parameters read:

$$\theta_0 = \eta_0, \quad \theta_1 = 1/2, \quad \theta_t = 2\eta_0, \quad \theta_\infty = -1/2.$$

Corresponding solutions of P_6 are

$$\begin{aligned} y_{12} &= \frac{(1-s^2)(\eta_0^2(2s+1)^2-1)}{\eta_0^2(2s+1)^3-1-2s^3}, \\ y_{21} &= \frac{1-s^2}{2s+1}, \end{aligned} \tag{3.1}$$

with the following associated functions

$$\sigma = -\frac{5}{8}\eta_0^2 - \eta_0^2 s + \frac{1}{16}s^2 \quad \text{and} \quad \tau = \frac{1}{(1-s^2)^{\frac{1}{16}}}\left(\frac{(1-s)^{\frac{13}{8}}}{(1+s)^{\frac{3}{8}}s^{\frac{5}{4}}}\right)^{\eta_0^2}.$$

By analogous arguments as at the end of Section 2 one finds that y_{21} solves the algebraic equation

$$\frac{1}{y_{21}^2} + \frac{4(t-1)}{(y_{21}-t)^2} = 0.$$

3.1.2 $RS_4^2(1+1+1|3|2+1)$

R-Transformation reads,

$$\mu = \rho \frac{\lambda(\lambda-1)(\lambda-t)}{(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \rho \frac{(\lambda-c)^3}{(\lambda-b)^2},$$

with

$$\rho = \frac{(1-s)(s+1)^3}{s^2+s+1}, \quad b = \frac{1}{(1-s)(s^2+s+1)}, \quad c = \frac{1}{1-s^2}, \text{ and } t = \frac{2s+1}{(1-s)(s+1)^3},$$

where $s \in \mathbb{C}$. To define the RS-transformation we choose η-parameters,

$$\eta_1 = \frac{1}{3}, \qquad \eta_\infty = -\frac{1}{2},$$

and arbitrary $\eta_0 \in \mathbb{C}$. Then RS-transformation has the following form,

$$\Psi(\lambda) = \left(\sqrt{\frac{\lambda-c}{\lambda-b}} J_c + \sqrt{\frac{\lambda-b}{\lambda-c}} J_b \right) \Phi(\mu),$$

where

$$J_b = \begin{pmatrix} 1 & -\dfrac{6\,\eta_0+1}{6\,\eta_0-5} \\ 0 & 0 \end{pmatrix}, \qquad J_c = J_b^*.$$

The function $\Psi(\lambda)$ has the following parameters of formal monodromy:

$$\theta_0 = \theta_1 = \theta_t = \eta_0, \quad \theta_\infty = -\frac{1}{2}.$$

We find

$$\begin{aligned} y_{12} &= \frac{(2s+1)(36\eta_0^2(s^2+s+1)^2 - (s^2-5s-5)^2)}{(s+1)((36\eta_0^2-25)(s^6+3s^5+3s+1) + 6(36\eta_0^2-7)(s^4+s^2) + (252\eta_0^2+41)s^3)}, \\ y_{21} &= \frac{(2s+1)}{(s+1)(s^2+s+1)}. \end{aligned}$$

and

$$\begin{aligned} \sigma(t) &= \frac{(108\eta_0^2+1)(s^4+2s^3+2s+1) + 6(36\eta_0^2-1)s^2}{288(s+1)^3(s-1)}, \\ \tau(t) &= \left((s+1)\Big(\frac{1}{s}+1\Big)\right)^{\frac{1}{24}} \left((s-1)\Big(\frac{1}{s}-1\Big)\right)^{\frac{\eta_0^2}{2}} \left((s+2)\Big(\frac{1}{s}+2\Big)\right)^{-\frac{1}{32}+\frac{5}{8}\eta_0^2}. \end{aligned}$$

Note, that the function $y_{21} = y_{21}(t)$ solves the following algebraic equation,

$$\frac{t(t-1)}{(y_{21}-t)^2} - \frac{(t-1)}{(y_{21}-1)^2} + \frac{t}{y_{21}^2} = 0.$$

Remark 2 *In the case $\eta_0 = 0$ solution $y_{12}(t)$ coincide with the so-called Tetrahedron Solution found in* [3, 4] [1]*:*

$$y(t) = \frac{(h-1)^2(1+3h)(9h^2-5)^2}{(1+h)(25-207h^2+1539h^4+243h^6)}, \quad t = -\frac{(h-1)^3(1+3h)}{(h+1)^3(1-3h)},$$

where parameter $h = s/(s+2)$. Clearly, it can be obtained by a Bäcklund transformation from a simpler solution $y_{12}(t)$ since the same is true for solution $y_{21}(t)$ with arbitrary η_0.

[1] There is a misprint in the sign of x in [4] p. 139 formula (A_3).

3.2 RS-transformations with fixed t

3.2.1 $RS_4^2(1+1+1|3|3)$

R-transformation reads:

$$\mu = \mp 3i\sqrt{3}\lambda(\lambda-1)(\lambda-t) \qquad \text{and} \qquad \mu - 1 = \mp 3i\sqrt{3}(\lambda-c)^3,$$

with

$$t = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \quad \text{and} \quad c = \frac{1}{2} \pm i\frac{\sqrt{3}}{6},$$

where, and thereafter in this subsection, one should take in all formulae either upper or lower signs correspondingly; so that we actually have two R-transformations.

For each R-transformation there are two different RS-transforms which correspond to the following choice of the η-parameters: $\eta_1 = 1/3$ or $\eta_1 = 2/3$, whereas η_0 and η_∞ are arbitrary complex numbers in both cases. Resulting θ-parameters are: $\theta_0 = \theta_1 = \theta_t = \eta_0$, in both cases, and $\theta_\infty = 3\eta_\infty + 1$ or $\theta_\infty = 3\eta_\infty$, correspondingly. We haven't checked yet whether one of these, formally different, RS-transformations can be obtained from the other by simply making the shift of $\eta_\infty \to \eta_\infty - 1/3$. Consider the first RS-transformation ($\eta_1 = 1/3$):

$$\Psi(\lambda) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \sqrt{\lambda - c} + \begin{pmatrix} 1 & -p \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{\lambda - c}} \right) \Phi(\mu), \qquad p = \frac{3\eta_0 - 3\eta_\infty - 1}{3\eta_0 + 3\eta_\infty - 1}. \tag{3.2}$$

The function $\Psi(\lambda)$ has the following parameters of formal monodromy:

$$\theta_0 = \theta_1 = \theta_t = \eta_0, \qquad \theta_\infty = 3\eta_\infty + 1. \tag{3.3}$$

The residue matrices of Eq. (1.2) read:

$$A_0 = \begin{pmatrix} -\frac{1}{2}\eta_\infty - \frac{1}{6} & \frac{6\eta_\infty p}{3 \pm i\sqrt{3}} \\ \frac{(3 \pm i\sqrt{3})q}{216\eta_\infty} & \frac{1}{2}\eta_\infty + \frac{1}{6} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -\frac{1}{2}\eta_\infty - \frac{1}{6} & -\frac{6\eta_\infty p}{3 \mp i\sqrt{3}} \\ -\frac{(3 \mp i\sqrt{3})q}{216\eta_\infty} & \frac{1}{2}\eta_\infty + \frac{1}{6} \end{pmatrix},$$

$$A_t = \begin{pmatrix} -\frac{1}{2}\eta_\infty - \frac{1}{6} & \pm i\eta_\infty\sqrt{3}p \\ \frac{\mp i\sqrt{3}q}{108\eta_\infty} & \frac{1}{2}\eta_\infty + \frac{1}{6} \end{pmatrix}, \qquad q = 9(\eta_0 + \eta_\infty)^2 - 1.$$

Using these formulae we find:

$$\begin{aligned} y_{12}\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) &= \frac{1}{2} \pm i\frac{\sqrt{3}}{6}, \qquad y_{12}'\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) = \frac{1}{3}, \\ y_{21}\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) &= \infty, \quad q \neq 0. \end{aligned}$$

In the case $q = 0$ the value of $y_{21}(1/2 \pm i\sqrt{3}/2)$ can't be determined.

It is worth to notice that making a special choice of the η-parameters in this RS-transformation $\eta_0 = 0$, $\eta_1 = 1/3$, and $\eta_\infty = 1$, we arrive to the solution considered by Manin (see p. 81 of [9]).

Indeed, in [9], a Frobenius manifold of dimension three is described by a solution of P6 with the initial data

$$y(t) = \frac{1}{2} + i\frac{\sqrt{3}}{6} \qquad \text{and} \qquad y'(t) = \frac{1}{3}$$

given at the point $t=\frac{1}{2}+\frac{\imath\sqrt{3}}{2}$ with the parameters

$$(\alpha,\beta,\gamma,\delta)=(\frac{9}{2},0,0,\frac{1}{2}).$$

For θ-parameters it means $\theta_0=\theta_1=\theta_t=0$ and θ_∞ is either -2 or 4. In the following construction $\theta_\infty=4$. We obtain these θ-parameters by putting $\eta_0=0$, $\eta_1=1/3$, and $\eta_\infty=1$ in Eq. (3.3).

Since, our construction allows us to find Ψ-function *explicitly*, see (3.2), we can write down the monodromy data. They are:

$$M_0=\begin{pmatrix} 1+i\sqrt{3} & -\frac{27}{8\pi^3}\Gamma(\frac{2}{3})^6(\sqrt{3}+i) \\ -\frac{2}{9}\frac{\pi^3(-i+\sqrt{3})}{\Gamma(\frac{2}{3})^6} & 1-i\sqrt{3}\end{pmatrix},$$

$$M_1=\begin{pmatrix} 1+7i\sqrt{3} & \frac{27}{8\pi^3}\Gamma(\frac{2}{3})^6(3\sqrt{3}-13i) \\ \frac{2}{9}\frac{\pi^3(3\sqrt{3}+13i)}{\Gamma(\frac{2}{3})^6} & 1-7i\sqrt{3}\end{pmatrix},$$

$$M_t=\begin{pmatrix} 1+i\sqrt{3} & \frac{27}{8\pi^3}\Gamma(2/3)^6(-i+\sqrt{3}) \\ \frac{2}{9}\frac{\pi^3(\sqrt{3}+i)}{\Gamma(\frac{2}{3})^6} & 1-i\sqrt{3}\end{pmatrix},$$

$$M_\infty=\begin{pmatrix} 1+3i\sqrt{3} & -\frac{81}{4\pi^3}i\Gamma(\frac{2}{3})^6 \\ \frac{4}{3}\frac{i\pi^3}{\Gamma(\frac{2}{3})^6} & 1-3i\sqrt{3}\end{pmatrix}.$$

These matrices M coincide with the corresponding monodromy matrices for the representation of the fundamental group defined in [12]. They satisfy the following cyclic relation: $M_\infty M_1 M_t M_0=1$. Note, that since θ_∞ is an integer number, M_∞ is not a diagonalizable matrix.

3.2.2 $RS_4^2(2+1|3|2+1)$

Corresponding R-transformation reads,

$$\mu=-\frac{\lambda(\lambda-1)^2}{3(\lambda-1/9)^2}\qquad\text{and}\qquad\mu-1=-\frac{(\lambda+1/3)^3}{3(\lambda-1/9)^2}.$$

Using this R-transformation one can define three different RS-transformations corresponding to the following following choice of the η-parameters:

1. η_0 and η_1 are arbitrary, $\eta_\infty=\frac{1}{2}$. The θ-parameters in Eq. (1.2) with $t=-1/3$ are as follows: $\theta_0=\eta_0$, $\theta_1=2\eta_0$, $\theta_t=3\eta_1$, and $\theta_\infty=-\frac{1}{2}$;

2. η_0 and η_∞ are arbitrary, $\eta_1=\frac{1}{3}$. The θ-parameters in Eq. (1.2) with $t=1/9$ are as follows: $\theta_0=\eta_0$, $\theta_1=2\eta_0$, $\theta_t=2\eta_\infty$, and $\theta_\infty=\eta_\infty-1$;

3. η_0 and η_∞ are arbitrary, $\eta_1=\frac{2}{3}$. The θ-parameters in Eq. (1.2) with $t=1/9$ are as follows: $\theta_0=\eta_0$, $\theta_1=2\eta_0$, $\theta_t=2\eta_\infty$, and $\theta_\infty=\eta_\infty$.

4 RS-transformations of the rank 4

In the case $r=4$ ($N_4=5$) the total number of different triples are $\frac{1}{6}\cdot5\cdot6\cdot7=35$. According to Eq. (1.7) twenty triples survive after the first stage , since $a_0=2$, $a_1=a_2=$

$a_4 = 1$, and $a_3 = 0$. As follows from Eqs. (1.8) and (1.9) two triples, $(1{+}1{+}1{+}1|2{+}2|2{+}2)$ and $(2{+}1{+}1|2{+}1{+}1|2{+}2)$, define RS-transformations with arbitrary parameter t which have an additional parameter; five triples, $(1+1+1+1|2+2|4)$, $(2+1+1|2+1+1|4)$, $(2+1+1|2+2|2+2)$, $(2+1+1|2+2|3+1)$, and $(2+1+1|3+1|3+1)$, define RS-transformations with arbitrary t; seven triples, $(1+1+1+1|4|4)$, $(2+1+1|2+2|4)$, $(2+1+1|3+1|4)$, $(2+2|2+2|2+2)$, $(2+2|2+2|3+1)$, $(2+2|3+1|3+1)$, and $(3{+}1|3{+}1|3{+}1)$, correspond to the RS-transformations with fixed t; finally, the following six triples, $(2+1+1|4|4)$, $(2+2|2+2|4)$, $(2+2|3+1|4)$, $(2+2|4|4)$, $(3+1|3+1|4)$, $(3+1|4|4)$, do not define any RS-transformation.

4.1 RS-Transformations with arbitrary t

4.1.1 $RS_4^2(3+1|3+1|2+1+1)$

R-Transformation reads,

$$\mu = \frac{\rho\lambda(\lambda-a)^3}{(\lambda-b)^2(\lambda-1)} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-t)(\lambda-c)^3}{(\lambda-b)^2(\lambda-1)},$$

where

$$\begin{aligned} \rho &= \frac{(1-2s)^3}{(1-3s^2)^2(1-3s)^2}, \qquad a = \frac{(1-3s^2)}{(1-2s)}(3s^2-2s+1), \\ b &= \frac{(1-3s^2)}{(1-3s)}(3s^2-3s+1), \qquad c = 1-3s^2, \end{aligned}$$

and

$$t = \frac{(1-3s^2)}{(1-2s)^3}(3s^2-3s+1)^2,$$

with arbitrary $s \in \mathrm{C}$. We consider below two different choices of the η-parameters, generating however equivalent seed RS-transformations, which are associated with this R-transformation. The reason, why we present here two equivalent constructions, is explained in Remark 3:

4.1.1.A. The first RS-transformation can be defined by making the following choice of the η-parameters in Eq. (1.1):

$$\eta_0 = 1/3, \qquad \eta_1 = 1/3, \qquad \eta_\infty = -1/2. \tag{4.1}$$

and reads,

$$\Psi(\lambda) = \left(J_1^*\sqrt{\frac{\lambda-b}{\lambda-1}} + J_b^*\sqrt{\frac{\lambda-1}{\lambda-b}}\right)\left(J_a\sqrt{\frac{\lambda-a}{\lambda-c}} + J_c\sqrt{\frac{\lambda-c}{\lambda-a}}\right)\Phi(\mu),$$

where

$$J_a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad J_c = J_a^*,$$

and

$$J_b = \frac{1}{6s(1-2s)}\begin{pmatrix} 1-3s^2 & -(1-3s)(s-1) \\ \frac{(1-3s)(1-3s^2)}{s-1} & -(1-3s)^2 \end{pmatrix}, \qquad J_1 = 1 - J_b.$$

It results in the following values of the θ-parameters in Eq. (1.2):

$$\theta_0 = 1/3, \quad \theta_1 = 1/2, \quad \theta_t = 1/3, \quad \theta_\infty = -1/2,$$

and generates the following solutions of P_6 and associated functions:

$$\begin{aligned} y_{12} &= -\frac{(3s^3-3s^2+3s-1)(3s^2-3s+1)(9s^5-15s^4-30s^3+60s^2-35s+7)}{(1-3s)(1-2s)(135s^6-378s^5+441s^4-288s^3+129s^2-42s+7)}, \\ y_{21} &= \frac{(3s^2-3s+1)(3s^2-2s+1)}{(1-3s)(1-2s)}, \end{aligned}$$

$$\begin{aligned} \sigma &= \frac{432s^6-972s^5+765s^4-176s^3-81s^2+54s-9}{144(1-2s)^3}, \\ \tau &= \frac{(3s-2)^{\frac{1}{16}}(1-2s)^{\frac{1}{12}}}{s^{\frac{3}{16}}(3s^2-3s+1)^{\frac{1}{18}}(1-3s^2)^{\frac{13}{144}}}. \end{aligned}$$

4.1.1.B. Another choice of the η-parameters is as follows:

$$\eta_0 = 1/3, \qquad \eta_1 = 2/3, \qquad \eta_\infty = -1/2. \tag{4.2}$$

However, making a proper Schlesinger transformation of Eq. (1.1) and further transformation which is related with the reflection, $\eta_\infty \to -\eta_\infty$ we see that corresponding RS-transformations and, hence, algebraic solutions are equivalent to the ones constructed in **A**. Below we present a bit different construction of the RS-transformation starting with the choice of the η-parameters given by (4.1); this, however, results in the same algebraic solutions as for the choice (4.2):

$$\Psi(\lambda) = \left(J_t^*\sqrt{\frac{\lambda-b}{\lambda-t}} + \hat{J}_b^*\sqrt{\frac{\lambda-t}{\lambda-b}}\right)\left(J_a\sqrt{\frac{\lambda-a}{\lambda-c}} + J_c\sqrt{\frac{\lambda-c}{\lambda-a}}\right)\Phi(\mu),$$

where μ, J_a, and J_c are the same as in **A**,

$$\hat{J}_b = \frac{1}{2s}\begin{pmatrix} -s+1 & -s+1 \\ -1+3s & -1+3s \end{pmatrix}, \qquad J_t = 1 - J_b.$$

The function Ψ solves Eq. (1.2) with the following values of the θ-parameters:

$$\theta_0 = 1/3, \quad \theta_1 = 1/2, \quad \theta_t = 2/3, \quad \theta_\infty = -1/2,$$

and generates the following solutions of P_6 and related functions:

$$\begin{aligned} y_{12} &= -\frac{(5s^2-4s+1)(3s^2-3s+1)(45s^4-102s^3+96s^2-42s+7)}{(135s^5-405s^4+450s^3-240s^2+63s-7)(1-2s)^2}, \\ y_{21} &= \frac{(s-1)(1-3s)(1-3s^2)(3s^2-3s+1)}{(9s^3-9s^2+3s-1)(1-2s)^2}. \end{aligned} \tag{4.3}$$

$$\begin{aligned} \sigma &= \frac{108s^6-216s^5+9s^4+244s^3-225s^2+90s-15}{144(1-2s)^3}, \\ \tau &= \frac{(1-2s)^{\frac{1}{3}}(1-3s^2)^{\frac{17}{144}}}{s^{\frac{5}{16}}(3s-2)^{\frac{1}{16}}(3s^2-3s+1)^{\frac{5}{36}}}. \end{aligned}$$

Remark 3 *It is interesting to note that for the θ-parameters considered in this subsection, i.e., $\theta_0 = 1/3$, $\theta_1 = 1/2$, $\theta_\infty = -1/2$, and $\theta_t = 1/3$ or $\theta_t = 2/3$, we have constructed here and in Subsection* 3.1.1 *two different algebraic solutions: given by Eqs.* (3.1) *and* (4.3). *Indeed, solution* (4.3) *has three finite poles at points $t_k = t(s_k)$, where $s_1 = 1/3 + \sqrt[3]{2}/3$ and $s_{2,3} = 1/3 - \sqrt[3]{2}(1 \pm i\sqrt{3})/6$ ($t_1 \approx 1.0577\ldots, t_{2,3} \approx 0.8391\ldots \pm i0.3357\ldots$), while* (3.1) *has only one pole at* $3/4$.

4.1.2 $RS_4^2(2+2|2+2|2+1+1)$

R-Transformation is as follows:

$$\mu = \frac{\rho\lambda^2(\lambda-1)^2}{(\lambda-t)(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-c_1)^2(\lambda-c_2)^2}{(\lambda-b)^2(\lambda-t)},$$

where

$$\rho = \frac{2s^2-1}{4(2s^2+2s+1)}, \quad b = \frac{s(2s+1)}{2s^2-1}, \quad c_1 = \frac{2s^2}{2s^2-1}, \quad c_2 = \frac{(2s+1)^2}{2s^2-1},$$

and

$$t = \frac{s^2(2s+1)^2}{(2s^2-1)(2s^2+2s+1)},$$

with $s \in \mathbb{C}$. For the choice: $\eta_0 \in \mathbb{C}$ is arbitrary, $\eta_1 = 1/2$, and $\eta_\infty = -1/2$, *RS*-transformation reads,

$$\Psi(\lambda) = \left(J_t^*\sqrt{\frac{\lambda-c_2}{\lambda-t}} + J_{c_2}^*\sqrt{\frac{\lambda-t}{\lambda-c_2}}\right)\left(J_b\sqrt{\frac{\lambda-b}{\lambda-c_1}} + J_{c_1}\sqrt{\frac{\lambda-c_1}{\lambda-b}}\right)\Phi(\mu),$$

where

$$J_{c_1} = J_{c_2} = \begin{pmatrix} 0 & \frac{\eta_0}{\eta_0-1} \\ 0 & 1 \end{pmatrix}, \quad J_b = J_{c_1}^*, \quad J_t = 1 - J_{c_2}.$$

Parameters:

$$\theta_0 = 2\eta_0, \quad \theta_1 = 2\eta_0, \quad \theta_t = 1/2, \quad \theta_\infty = -1/2.$$

Solutions of P_6 and related functions σ and τ are as follows:

$$\begin{aligned} y_{12} &= \frac{s(2s+1)(4\eta_0^2(2s^2+2s+1)^2 - s^2(2s+1)^2)}{(2s^2+2s+1)((4\eta_0^2-1)(2s^2+2s+1)^2 + 3s(s+1)(2s+1))}, \\ y_{21} &= \frac{s(2s+1)}{2s^2+2s+1}. \end{aligned}$$

$$\begin{aligned} \sigma &= -\frac{(2s^2+2s+1)\eta_0^2}{2s^2-1}, \\ \tau &= \left(\frac{(2s^2-1)^2}{s(s+1)(2s+1)}\right)^{2\eta_0^2}. \end{aligned}$$

Remark 4 *The function y_{21} solves the following algebraic equation,*

$$\frac{t}{y_{21}^2} - \frac{(t-1)}{(y_{21}-1)^2} = 0,$$

and therefore, can be written as $y_{21} = t - \sqrt{t^2 - t}$. Thus the solutions constructed in this subsection are not new, they coincide with the ones obtained in Section 2 in the case $\eta_\infty = -1/2$ and $\eta_0 \to 2\eta_0$. The explanation of this fact is that corresponding RS-transformation is actually a combination of a quadratic transformation for Eq. (1.1) *with the quadratic transformation obtained in Section* 2.

Remark 5 *In the case* $\eta_0 = 1/6$ θ*-parameters of solutions constructed in this subsection are as follows:* $\theta_0 = 1/3$, $\theta_1 = 1/3$, $\theta_t = 1/2$, $\theta_\infty = -1/2$. *Therefore, by interchanging points* 1 *and* t *in Eq. (1.2), we can construct the solutions of* P_6 *for the same case:* $\theta_0 = 1/3$, $\theta_1 = 1/2$, $\theta_t = 1/3$, $\theta_\infty = -1/2$ *as in the previous subsection. Consider this in more detail to check that these solutions are different from the ones constructed previously.*

Transformation reads:

$$\hat{t} = 1/t, \qquad \hat{\lambda} = \lambda/t, \qquad \hat{\Psi}(\hat{\lambda}) = \hat{t}^{-\frac{\theta_\infty}{2}\sigma_3}\Psi(\hat{\lambda}/\hat{t}),$$

$$\hat{\theta}_0 = \theta_0 = 2\eta_0, \quad \hat{\theta}_1 = \theta_t = 1/2, \quad \hat{\theta}_t = \theta_1 = 2\eta_0, \quad \hat{\theta}_\infty = \theta_\infty = -1/2.$$

This transformation generates the following solutions of P_6 *and related functions* σ *and* τ*, with* θ*-parameters changed to* $\hat{\theta}$ *parameters and* t *to* $\hat{t} = 1/t$*:*

$$\begin{aligned} \hat{y}_{12} &= \frac{(2s^2-1)(4\eta_0^2(2s^2+2s+1)^2 - s^2(2s+1)^2)}{s(2s+1)((4\eta_0^2-1)(2s^2+2s+1)^2 + 3s(s+1)(2s+1))}, \\ \hat{y}_{21} &= \frac{2s^2-1}{s(2s+1)} = 1 - \sqrt{1-\hat{t}}, \end{aligned}$$

$$\begin{aligned} \hat{\sigma} &= \frac{(s+1)^2 - 16\eta_0^2 s(2s+1)(2s^2+3s+2)}{16s^2(2s+1)^2}, \\ \hat{\tau} &= \hat{t}^{-\frac{1}{16}}\left(\frac{(2s^2-1)^3}{(s+1)^2(2s^2+2s+1)}\right)^{\eta_0^2}, \quad \text{where} \quad \hat{t} = \frac{(2s^2-1)(2s^2+2s+1)}{s^2(2s+1)^2}. \end{aligned}$$

The parameter s *can be excluded from the above formulae via the quadratic equation,* $s(2s+1)\sqrt{1-\hat{t}} = (s+1)$.

4.1.3 $RS_4^2(2+1+1|2+2|2+1+1)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda-1)(\lambda-a)^2}{(\lambda-t)(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-c_1)^2(\lambda-c_2)^2}{(\lambda-t)(\lambda-b)^2},$$

where

$$\begin{aligned} \rho &= \frac{(s-1)^2(-s^2+c_1s^2+c_1)}{(s+1)^2(2c_1^2s^2 - 3c_1s^2 + s^2 - 4sc_1^2 + 4c_1s + 2c_1^2 - c_1)}, \\ a &= \frac{s(c_1s^3 - s^3 - 2c_1^2s^2 + 2c_1s^2 + 4sc_1^2 - 3c_1s - 2c_1^2)}{(s-1)^2(-s^2+c_1s^2+c_1)}, \quad b = \frac{s^2}{s^2-1}, \\ c_2 &= \frac{s^2(c_1s^2 - 2c_1s + 2s + c_1 - s^2)}{2c_1s^2 + 2s^3 - s^4 - s^2 - 2c_1s + c_1s^4 - 2c_1s^3 + c_1}, \end{aligned}$$

and

$$t = \frac{(c_1s^2 - 2c_1s + 2s + c_1 - s^2)^2c_1^2}{(2c_1^2s^2 - 3c_1s^2 + s^2 - 4sc_1^2 + 4c_1s + 2c_1^2 - c_1)(-s^2+c_1s^2+c_1)},$$

with the parameters s and $c_1 \in \mathbb{C}$. For the choice of η-parameters: $\eta_0 = 1/2$, $\eta_1 = 1/2$, and $\eta_\infty = -1/2$, RS-transformation can be written as follows

$$\Psi(\lambda) = \left(J_a \sqrt{\frac{\lambda - a}{\lambda - c_2}} + J_{c_2} \sqrt{\frac{\lambda - c_2}{\lambda - a}} \right) \left(J_b \sqrt{\frac{\lambda - b}{\lambda - c_1}} + J_{c_1} \sqrt{\frac{\lambda - c_1}{\lambda - b}} \right) \Phi(\mu),$$

where

$$\begin{aligned} J_{c_1} &= \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad J_b = J_{c_1}^*, \qquad J_a = 1 - J_{c_2}, \\ J_{c_2} &= \frac{1}{4s(c_1 s - s - c_1)} \begin{pmatrix} (c_1 s^2 - s^2 - c_1)(s+1) & c_1(s-1)^3 + s^2(3-s) \\ -(c_1 s^2 - s^2 - c_1)(s+1) & -c_1(s-1)^3 - s^2(3-s) \end{pmatrix}. \end{aligned}$$

The function Ψ solves Eq. (1.2) with the following parameters:

$$\theta_0 = 1/2, \quad \theta_1 = 1/2, \quad \theta_t = 1/2, \quad \theta_\infty = -1/2.$$

Solutions of P_6 and related functions are:

$$\begin{aligned} y_{12} &= \frac{c_1(c_1(s-1)^2 + 2s - s^2)(2c_1^2(s-1)^2 + c_1 + 4c_1 s - c_1 s^2 - s^2)}{3(2c_1^2(s-1)^2 - 3c_1 s^2 + s^2 + 4c_1 s - c_1)(-s^2 + c_1 s^2 + c_1)} = t - \frac{1}{3}\sqrt{t(t-1)}, \\ y_{21} &= \frac{c_1(c_1(s-1)^2 + 2s - s^2)}{-s^2 + c_1 s^2 + c_1} = t + \sqrt{t(t-1)}, \end{aligned}$$

$$\begin{aligned} \sigma &= -\frac{-s^2 + c_1 s^2 + c_1}{16(2c_1^2(s-1)^2 - 3c_1 s^2 + s^2 + 4c_1 s - c_1)} = -\frac{1}{16}\,\frac{t - \sqrt{t(t-1)}}{t + \sqrt{t(t-1)}}, \\ \tau &= \frac{(2c_1^2(s-1)^2 - 3c_1 s^2 + s^2 + 4c_1 s - c_1)^{\frac{1}{4}}}{(c_1(c_1-1)(c_1(s-1)^2 - s^2)(c_1(s-1)^2 + 2s - s^2))^{\frac{1}{8}}} = \left(\frac{(\sqrt{t} + \sqrt{t-1})^2}{\sqrt{t}\,\sqrt{t-1}} \right)^{\frac{1}{8}}. \end{aligned}$$

The branches of the square roots are chosen such that, $\sqrt{t}\sqrt{t-1} = \sqrt{t(t-1)}$.

Remark 6 *This RS-transformation is a combination of two quadratic transformations. Corresponding solutions of P_6 are the special case of those obtained in Section 2 for $\eta_0 = -\eta_\infty = 1/2$.*

4.1.4 $RS_4^2(1+1+1+1|2+2|4)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda-1)(\lambda-t)}{(\lambda-b)^4} \qquad \text{and} \qquad \mu - 1 = -\frac{(\lambda-c_1)^2(\lambda-c_2)^2}{(\lambda-b)^4},$$

where

$$\rho = \frac{(s^2-2)^2}{(s+1)^2}, \quad b = \frac{s(s+2)}{2(s+1)}, \quad c_1 = \frac{s^2}{2}, \quad c_2 = \frac{(s+2)^2}{2(s+1)^2},$$

and

$$t = \frac{s^2(s+2)^2}{4(s+1)^2}.$$

With this R-transformation one can associate two different seed RS-transformations:
4.1.4.A The first RS-transformation can be defined by making the following choice of the η-parameters: $\eta_0 \in \mathbb{C}$ is arbitrary, $\eta_1 = 1/2$, $\eta_\infty = -1/4$. Corresponding RS-transformation can be written as follows,

$$\Psi(\lambda) = \left(J_t^* \sqrt{\frac{\lambda - c_1}{\lambda - t}} + J_{c_1}^* \sqrt{\frac{\lambda - t}{\lambda - c_1}}\right) \left(J_b \sqrt{\frac{\lambda - b}{\lambda - c_2}} + J_{c_2} \sqrt{\frac{\lambda - c_2}{\lambda - b}}\right) T^{-1} \Phi(\mu),$$

where

$$T = \begin{pmatrix} \frac{(4\eta_0+3)(4\eta_0-1)}{16\eta_0} & 1 \\ \frac{(4\eta_0-3)(4\eta_0+1)}{16\eta_0} & 1 \end{pmatrix},$$

and

$$\begin{aligned} J_b &= \begin{pmatrix} \frac{4\eta_0-1}{8\eta_0} & -\frac{2}{4\eta_0-3} \\ -\frac{(4\eta_0-3)(4\eta_0-1)(4\eta_0+1)}{128\eta_0^2} & \frac{4\eta_0+1}{8\eta_0} \end{pmatrix}, \qquad J_{c_2} = J_b^*, \\ J_{c_1} &= \begin{pmatrix} -\frac{4\eta_0+1}{4s\eta_0} & \frac{4(4\eta_0 s+4\eta_0+1)}{s(4\eta_0-3)(4\eta_0-1)} \\ -\frac{(4\eta_0-3)(4\eta_0-1)(4\eta_0+1)}{64s\eta_0^2} & \frac{4\eta_0 s+4\eta_0+1}{4s\eta_0} \end{pmatrix}, \qquad J_t = 1 - J_{c_1}. \end{aligned}$$

The function Ψ solves Eq. (1.2) with the parameters:

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_0 + 1, \quad \theta_\infty = \eta_0.$$

Corresponding solutions of P_6 and related functions are as follows:

$$\begin{aligned} y_{12} &= -\frac{s(s+2)(-s^2+8\eta_0 s+8\eta_0+2)}{2(s+1)(-3s^2+8\eta_0 s-8s-2+8\eta_0)}, \\ y_{21} &= -\frac{s(s+2)(s^2+4\eta_0 s^2+8\eta_0 s-2)}{2(s+1)(3s^2+4\eta_0 s^2+8s+8\eta_0 s+2)}, \\ \sigma &= \frac{32(s+1)^2\eta_0^2 - 16(s+1)(s^2+s-1)\eta_0 - s^4 - 8s^3 - 12s^2 + 4}{64(s+1)^2}, \\ \tau &= \frac{(s+1)^{\frac{1}{8}}(s^2+4s+2)^{\frac{1}{8}+\frac{3}{4}\eta_0+\frac{1}{2}\eta_0^2}}{[s(s+2)]^{\frac{1}{8}+\frac{1}{2}\eta_0+\eta_0^2}(s^2-2)^{\frac{1}{8}+\frac{1}{4}\eta_0-\frac{1}{2}\eta_0^2}} \\ &= \left(\frac{\sqrt{t}+1}{2\sqrt{t}(\sqrt{t}-1)}\right)^{\frac{1}{8}} \left(\frac{(\sqrt{t}+1)^3}{t(\sqrt{t}-1)}\right)^{\frac{\eta_0}{4}} \left(1 - \frac{1}{t}\right)^{\frac{\eta_0^2}{2}}. \end{aligned}$$

Remark 7 *We present here also an equivalent RS-transformation,*

$$\Psi(\lambda) = \left(J_+^* \sqrt{\lambda - c_1} + J_{+,\,c_1}^* \frac{1}{\sqrt{\lambda - t}}\right) \left(J_b \sqrt{\frac{\lambda - b}{\lambda - c_2}} + J_{c_2} \sqrt{\frac{\lambda - c_2}{\lambda - b}}\right) T^{-1} \Phi(\mu),$$

where the functions, $t = t(s)$ and $\lambda = \lambda(\mu)$, matrices, $\Phi(\mu)$, T, J_b, and J_{c_2}, are the same as above, and

$$J_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{+,\,c_1} = \begin{pmatrix} \frac{(s^2-2)(4\eta_0+1)(4\eta_0 s^2+4\eta_0 s+3s^2+4s+2)}{64\eta_0(\eta_0+1)(s+1)^2} & \frac{16\eta_0}{(4\eta_0-3)(4\eta_0-1)} \\ \frac{(4\eta_0-3)(16\eta_0^2-1)(s^2-2)(4\eta_0 s^2+4\eta_0 s+3s^2+4s+2)}{1024\eta_0^2(\eta_0+1)(s+1)^2} & 1 \end{pmatrix}.$$

This function $\Psi(\lambda)$ solves Eq. (1.2) *with the parameters:*

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_0, \quad \theta_\infty = \eta_0 + 1,$$

and generates the following solutions of P_6 and related functions:

$$\begin{aligned}
y_{12} &= -\frac{s(s+2)}{2(s+1)} = -\sqrt{t}, \\
y_{21} &= -\frac{s(s+2)(32s(s+1)(s+2)\eta_0^2 + 4(s^2+4s+2)^2\eta_0 + (s^2-2)^2)}{2(s+1)(32s(s+1)(s+2)\eta_0^2 + 4(s^2+4s+2)^2(3\eta_0+2) + (s^2-2)^2)}, \\
\sigma &= \frac{32(s+1)^2\eta_0^2 + 16(s+1)(s^2+3s+1)\eta_0 - s^4 + 12s^2 + 16s + 4}{64(s+1)^2}, \\
\tau &= \frac{(s+1)^{\frac{1}{8}}(s^2-2)^{\frac{1}{8}+\frac{3}{4}\eta_0+\frac{1}{2}\eta_0^2}}{[s(s+2)]^{\frac{1}{8}+\frac{1}{2}\eta_0+\eta_0^2}(s^2+4s+2)^{\frac{1}{8}+\frac{1}{4}\eta_0-\frac{1}{2}\eta_0^2}} \\
&= \left(\frac{\sqrt{t}-1}{2\sqrt{t}(\sqrt{t}+1)}\right)^{\frac{1}{8}} \left(\frac{(\sqrt{t}-1)^3}{t(\sqrt{t}+1)}\right)^{\frac{\eta_0}{4}} \left(1-\frac{1}{t}\right)^{\frac{\eta_0^2}{2}}.
\end{aligned}$$

4.1.4.B The second RS-transformation is defined by the following choice of the η-parameters: $\eta_0 \in \mathbb{C}$ is arbitrary, $\eta_1 = -\eta_\infty = 1/2$. Corresponding RS-transformation reads,

$$\Psi(\lambda) = \left(J_b\sqrt{\frac{\lambda-b}{\lambda-c_1}} + J_{c_1}\sqrt{\frac{\lambda-c_1}{\lambda-b}}\right)\left(J_b\sqrt{\frac{\lambda-b}{\lambda-c_2}} + J_{c_2}\sqrt{\frac{\lambda-c_2}{\lambda-b}}\right) T^{-1}\Phi(\mu).$$

In this case the residue matrices in Eq. (1.2) are as follows:

$$A_0 = \frac{\eta_0}{2}\sigma_3, \quad A_1 = -\frac{\eta_0}{2}\sigma_3, \quad A_t = -\frac{\eta_0}{2}\sigma_3,$$

therefore

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_0, \quad \theta_\infty = \eta_0.$$

In this case solutions of P_6 are not defined, the functions σ and τ are very simple:

$$\sigma = \frac{1}{2}\eta_0^2, \qquad \tau = \frac{(-2+s^2)^{\frac{\eta_0^2}{2}}(s^2+4s+2)^{\frac{\eta_0^2}{2}}}{s^{\eta_0^2}(s+2)^{\eta_0^2}} = \left(\frac{t-1}{t}\right)^{\frac{\eta_0^2}{2}}.$$

4.1.5 $RS_4^2(1+1+1+1|2+2|2+2)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda-1)(\lambda-t)}{(\lambda-b_1)^2(\lambda-b_2)^2} \qquad \text{and} \qquad \mu - 1 = -\frac{(\lambda-c_1)^2(\lambda-c_2)^2}{(\lambda-b_1)^2(\lambda-b_2)^2},$$

where

$$\rho = \frac{4(c_1^2+b_2^2-c_1-b_2)}{c_1-b_2}, \quad b_1 = \frac{c_1(c_1+b_2-2)}{b_2-c_1}, \quad c_2 = -\frac{b_2(c_1+b_2-2)}{b_2-c_1},$$

$$t = \frac{(-2+c_1+b_2)b_2c_1}{c_1^2+b_2^2-c_1-b_2},$$

and c_1 and b_2 are parameters. We present below a seed RS-transformation corresponding to the following choice of the η-parameters: $\eta_0 \in \mathbb{C}$ is arbitrary, $\eta_1 = -\eta_\infty = 1/2$;

$$\Psi(\lambda) = \left(J_b\sqrt{\frac{\lambda - b_1}{\lambda - c_1}} + J_c\sqrt{\frac{\lambda - c_1}{\lambda - b_1}}\right)\left(J_b\sqrt{\frac{\lambda - b_2}{\lambda - c_2}} + J_c\sqrt{\frac{\lambda - c_2}{\lambda - b_2}}\right)T^{-1}\Phi(\mu).$$

Here

$$T = \begin{pmatrix} \frac{1}{2}\eta_0 + \frac{1}{2} & 1 \\ \frac{1}{2}\eta_0 - \frac{1}{2} & 1 \end{pmatrix}, \quad J_b = \frac{1}{2}\begin{pmatrix} 1 & -\frac{2}{\eta_0 - 1} \\ -\frac{1}{2}\eta_0 + \frac{1}{2} & 1 \end{pmatrix}, \quad J_c = J_b^*,$$

Parameters:

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_0, \quad \theta_\infty = \eta_0.$$

In this case all matrices A_p, $p = 0, 1, t$ are diagonal:

$$A_0 = A_1 = -A_t = -\frac{\eta_0}{2}\sigma_3,$$

so there is no solution to y. The functions σ and τ are very simple:

$$\sigma = -\frac{\eta_0^2}{2}\frac{(b_2 + c_1 - 1)(2b_2c_1 - c_1 - b_2)}{c_1^2 + b_2^2 - c_1 - b_2} = \frac{\eta_0^2}{2}(1 - 2t).$$

$$\tau = \left(\frac{(c_1^2 + b_2^2 - c_1 - b_2)^2}{b_2c_1(b_2 - 1)(c_1 - 1)(b_2 + c_1)(-2 + c_1 + b_2)}\right)^{\frac{\eta_0^2}{2}} = [t(t-1)]^{-\frac{\eta_0^2}{2}}.$$

4.1.6 $RS_4^2(2+1+1|4|2+1+1)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda - 1)(\lambda - a)^2}{(\lambda - t)(\lambda - b)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda - c)^4}{(\lambda - t)(\lambda - b)^2},$$

where

$$\begin{aligned} \rho &= \frac{(s^2 - 4s + 2)(s^2 - 2s + 2)^2}{(s^2 - 2)(3s^2 - 4s + 2)^2}, \quad a = -\frac{s^4(3s^2 - 8s + 6)}{(s^2 - 4s + 2)(s^2 - 2s + 2)^2}, \\ b &= -\frac{s^4}{(s^2 - 4s + 2)(3s^2 - 4s + 2)}, \quad c = \frac{s^3(s - 2)}{s^4 - 6s^3 + 12s^2 - 12s + 4}, \end{aligned}$$

$$t = \frac{s^4(s - 2)^4}{(s^2 - 2s + 2)^2(s^2 - 4s + 2)(s^2 - 2)},$$

with arbitrary $s \in \mathbb{C}$.There are two different seed RS-transformations which can be associated with this R-transformation:

4.1.6.A The first transformation is defined by the following choice of the η-parameters:

$$\eta_0 = \frac{1}{2}, \quad \eta_1 = \frac{1}{2}, \quad \text{and} \quad \eta_\infty = -\frac{1}{2}.$$

RS-transformation reads,

$$\Psi(\lambda) = \left(1 - \frac{J_c}{\lambda - c}\right)\left(J_b\sqrt{\frac{\lambda - b}{\lambda - a}} + J_a\sqrt{\frac{\lambda - a}{\lambda - b}}\right)\Phi(\mu),$$

where

$$J_b=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix},\quad J_a=J_b^*,$$

and

$$J_c=j\begin{pmatrix}1 & -\frac{s^2-2}{3s^2-4s+2}\\ \frac{3s^2-4s+2}{s^2-2} & -1\end{pmatrix},\quad j=-\frac{s^2(s-1)^2(s^2-2)}{(s^2-4s+2)(s^2-2s+2)^2}.$$

Parameters:

$$\theta_0=\frac{1}{2},\quad \theta_1=\frac{1}{2},\quad \theta_t=\frac{1}{2},\quad \text{and}\quad \theta_\infty=-\frac{1}{2}.$$

Corresponding solutions of P_6 and functions σ and τ are as follows:

$$\begin{aligned}
y_{12} &= \frac{s^2(s-2)^2(3s^4-12s^3+16s^2-8s+4)}{3(s^2-2s+2)^2(s^2-4s+2)(s^2-2)}=t+\frac{1}{3}\sqrt{t(t-1)},\\
y_{21} &= \frac{s^2(s-2)^2}{(s^2-2s+2)^2}=t-\sqrt{t(t-1)},\\
\sigma &= -\frac{(s^2-2s+2)^2}{16(s^2-2)(s^2-4s+2)}=-\frac{1}{16}\,\frac{t+\sqrt{t(t-1)}}{t-\sqrt{t(t-1)}},\\
\tau &= \frac{(s^2-2)^{\frac14}(s^2-4s+2)^{\frac14}}{s^{\frac14}(s-1)^{\frac14}(s-2)^{\frac14}}=\frac{(2\sqrt{t}-2\sqrt{t-1})^{\frac14}}{[t(t-1)]^{\frac{1}{16}}}.
\end{aligned}$$

Remark 8 *Note that solutions constructed in this subsection coincide with the ones obtained in Subsection* 4.1.3, *see also Remark* 6.

4.1.6.B We define another seed RS-transformation by the following choice of the η-parameters:

$$\eta_0=\frac{1}{2},\quad \eta_1=\frac{1}{4},\quad \text{and}\quad \eta_\infty=-\frac{1}{2}.$$

RS-transformation reads,

$$\Psi(\lambda)=\left(J_c^*\sqrt{\frac{\lambda-t}{\lambda-c}}+J_t^*\sqrt{\frac{\lambda-c}{\lambda-t}}\right)\left(J_b\sqrt{\frac{\lambda-b}{\lambda-a}}+J_a\sqrt{\frac{\lambda-a}{\lambda-b}}\right)\Phi(\mu).$$

Here

$$J_b=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix},\quad J_a=J_b^*,\qquad J_c=\begin{pmatrix}0 & -\frac{s^2-4s+6}{3s^2-4s+2}\\ 0 & 1\end{pmatrix},\quad J_t=1-J_c.$$

Parameters:

$$\theta_0=\frac{1}{2},\quad \theta_1=\frac{1}{2},\quad \theta_t=\frac{1}{2},\quad \theta_\infty=-\frac{1}{2}.$$

Solutions of P_6 and functions σ and τ are as follows:

$$\begin{aligned}
y_{21} &= \frac{s^3(s-2)(3s^2-8s+6)}{(s^2-2s+2)(3s^2-4s+2)(s^2-2)},\\
y_{12} &= y_{21}\frac{(3s^2-4s+2)(7s^6-44s^5+106s^4-112s^3+20s^2+80s-72)}{3(s^2-4s+2)(7s^6-36s^5+86s^4-112s^3+100s^2-80s+40)},\\
\sigma &= \frac{s^6-24s^5+122s^4-288s^3+364s^2-224s+56}{64(s^2-2s+2)^2(s^2-4s+2)},\\
\tau &= \frac{(s-2)^{\frac{1}{16}}(s^2-4s+2)^{\frac14}(s^2-2s+2)^{\frac18}}{s^{\frac{7}{16}}(s-1)^{\frac{7}{16}}}.
\end{aligned}$$

Remark 9 *In terms of t parameter s reads,*

$$s = 1 + \sqrt[4]{\frac{t}{t-1}} + \sqrt{1 + \sqrt{\frac{t}{t-1}}}.$$

4.1.7 $RS_4^2(2+1+1|2+2|3+1)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda-1)(\lambda-t)^2}{(\lambda-b)^3} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-c_1)^2(\lambda-c_2)^2}{(\lambda-b)^3}. \tag{4.4}$$

where

$$\begin{aligned} \rho &= \frac{s^3(s^2+1)^3}{(s^4+1)^3}, & b = \frac{(s+1)^4(s^2-s+1)^2}{4s(s^2+1)(s^4+1)}, \\ c_1 &= \frac{(s+1)^2(s^2-s+1)^2}{4s^3}, & c_2 = \frac{(s+1)^4(s^2-s+1)}{2(s^2+1)^3}, \end{aligned}$$

and

$$t = -\frac{(s+1)^4(s^2-s+1)^2(s^4-2s^3-2s+1)}{4s^3(s^2+1)^3}.$$

With this R-transformation one can associate two different seed RS-transformations: **4.1.7.A** The first seed RS-transformation can be associated with the following choice of the η-parameters:

$$\eta_0 \in \mathbb{C}, \quad \eta_1 = \frac{1}{2}, \quad \eta_\infty = -\frac{1}{3}.$$

Corresponding RS-transformation reads,

$$\Psi(\lambda) = \left(J_{c_1}^* \sqrt{\frac{\lambda-1}{\lambda-c_1}} + J_1^* \sqrt{\frac{\lambda-c_1}{\lambda-1}}\right)\left(J_b \sqrt{\frac{\lambda-b}{\lambda-c_2}} + J_{c_2}\sqrt{\frac{\lambda-c_2}{\lambda-b}}\right)\Phi(\mu).$$

Here

$$J_b = \begin{pmatrix} 1 & -\frac{6\eta_0-1}{6\eta_0-5} \\ 0 & 0 \end{pmatrix}, \quad J_{c_2} = J_b, \qquad J_{c_1} = \begin{pmatrix} j & \frac{(1-j)(6\eta_0-1)}{6\eta_0-5} \\ \frac{(6\eta_0-5)j}{6\eta_0-1} & 1-j \end{pmatrix}, \quad J_1 = 1 - J_{c_1},$$

$$j = -\frac{s(s^4+1)(6\eta_0-1)}{2(s^2+s+1)(s^2+1)^2}.$$

Parameters:

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0 - 1, \quad \theta_t = 2\eta_0, \quad \theta_\infty = \eta_\infty.$$

Solutions:

$$\begin{aligned} y_{12} = & -\frac{(s+1)^2(s^2-s+1)(s^4-2s^3-2s+1)}{2s^2(s^2+1)^2} \times \\ & \frac{(6s(s^2+1)(s^4+1)\eta_0 + (s^2+1)^4 + 4s^4)(6(s^4+1)^2\eta_0 - (s^2+1)^4 - 4s^4)}{(36(s^4+1)^4\eta_0^2 + 48s(s^2+1)(s-1)^2(s^4+1)^2(s^2+s+1)\eta_0 + Q(s))}, \end{aligned}$$

$$
\begin{aligned}
Q(s) = & -9s^{16} - 8s^{15} + 40s^{14} - 40s^{13} + 28s^{12} - 120s^{11} + 56s^{10} - 152s^9 + \\
& +10s^8 - 152s^7 + 56s^6 - 120s^5 + 28s^4 - 40s^3 + 40s^2 - 8s - 9, \\
y_{21} = & -\frac{(s+1)^2(s^4+1)(s^2-s+1)(s^4-2s^3-2s+1)(6s(s^2+1)\eta_0+s^4+1)}{2s^2(s^2+1)^2(6(s^4+1)^2\eta_0+s^8+4s^7-4s^6+4s^5-6s^4+4s^3-4s^2+4s+1)}, \\
\sigma = & -\frac{3(s^4+1)}{4s(s^2+1)}\eta_0^2 - \frac{s^8-s^7-s^6-s^5-s^3-s^2-s+1}{4s^2(s^2+1)^2} + \\
& \frac{4s^{12}+6s^{11}-15s^{10}-18s^8-6s^7-30s^6-6s^5-18s^4-15s^2+6s+4}{144s^3(s^2+1)^3} \\
\tau = & \frac{s^{\frac{1}{3}}(s^2+1)^{\frac{1}{3}}(s^4+2s^3+2s+1)^{\frac{1}{9}-\frac{5}{4}\eta_0+\frac{3}{2}\eta_0^2}\left((s^2+1)^2-s^2\right)^{-\frac{11}{72}+\frac{1}{2}\eta_0-\frac{3}{2}\eta_0^2}}{(s^4-2s^3-2s+1)^{\frac{5}{36}-\frac{1}{4}\eta_0-\frac{3}{2}\eta_0^2}(s^2-1)^{\frac{11}{36}-\eta_0+3\eta_0^2}}.
\end{aligned}
$$

4.1.7.B To associate another seed RS-transformation with R-transformation (4.4) we exchange notation: $t \longleftrightarrow b$, so that now:

$$
t = \frac{(s+1)^4(s^2-s+1)^2}{4s(s^2+1)(s^4+1)}, \quad b = -\frac{(s+1)^4(s^2-s+1)^2(s^4-2s^3-2s+1)}{4s^3(s^2+1)^3}.
$$

and the other parameters in (4.4) remain unchanged. RS-transformation can be defined by the following choice of parameters:

$$
\eta_0 = \eta_1 = \frac{1}{2},
$$

and arbitrary $\eta_\infty \in \mathbb{C}$. It reads,

$$
\Psi(\lambda) = \left(J_{c_1}^*\sqrt{\frac{\lambda-1}{\lambda-c_1}} + J_1^*\sqrt{\frac{\lambda-c_1}{\lambda-1}}\right)\left(J_a\sqrt{\frac{\lambda-a}{\lambda-c_2}} + J_{c_2}\sqrt{\frac{\lambda-c_2}{\lambda-a}}\right)\Phi(\mu).
$$

Here

$$
J_a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad J_{c_2} = J_a^*, \quad J_{c_1} = J_{c_2}, \quad J_1 = 1 - J_{c_1}.
$$

Parameters:

$$
\theta_0 = \frac{1}{2}, \quad \theta_1 = \frac{1}{2}, \quad \theta_t = 3\eta_\infty, \quad \theta_\infty = \eta_\infty.
$$

Corresponding solutions of P_6 and functions σ and τ are as follows:

$$
\begin{aligned}
y_{12} &= -\frac{(s+1)^2(s^2-s+1)(2(s^4-2s^3-2s+1)\eta_\infty+s^4+s^3+s+1)}{4(\eta_\infty-1)s(s^2+1)(s^4+1)}, \\
y_{21} &= -\frac{(s+1)^2(s^2-s+1)(2(s^4-2s^3-2s+1)\eta_\infty-s^4-s^3-s^2-s-1)}{4(\eta_\infty+1)s(s^2+1)(s^4+1)}, \\
\sigma &= -\frac{\eta_\infty^2(s^4+2s^3+4s^2+2s+1)(s^4-2s^3+4s^2-2s+1)}{4s(s^2+1)(s^4+1)}, \\
\tau &= \left(\frac{s(s^2+1)(s^4+1)^4}{(s^2-1)^5(s^2+s+1)^{\frac{5}{2}}(s^2-s+1)^{\frac{5}{2}}}\right)^{\eta_\infty^2}.
\end{aligned}
$$

4.2 RS-transformations with fixed t

One proves that the triple $(3+1|2+2|2+2)$ does not define any R-transformation; therefore only six triples of those seven mentioned in the beginning of section 4 define RS-transformations with fixed t.

4.2.1 $RS_4^2(2+1+1|2+2|4)$

4.2.1.A R-Transformation reads,

$$\mu = \frac{\lambda(\lambda-1)(\lambda-a)^2}{(\lambda-b)^4} \qquad \text{and} \qquad \mu - 1 = -\frac{(\lambda-t)^2}{2(\lambda-b)^4},$$

where

$$a = 2b - \frac{1}{2}, \quad b = \frac{1}{2} \pm \frac{\sqrt{2}}{4}, \quad t = \frac{3}{2}b - \frac{1}{4}.$$

To construct RS-transformation one chooses η-parameters as follows: $\eta_0 = \frac{1}{2}$, $\eta_\infty = \frac{1}{2}$ (or $\frac{1}{4}$), and $\eta_1 \in \mathbb{C}$ is arbitrary. This allows one to map Eq. (1.1) into Eq. (1.2) with the parameters:

$$\theta_0 = \frac{1}{2}, \quad \theta_1 = \frac{1}{2}, \quad \theta_t = 2\eta_1, \quad \theta_\infty = 2\eta_1, \quad \text{and} \quad t = \frac{1}{2} \pm \frac{3\sqrt{2}}{8}.$$

It can be presented as a superposition of two RS-transformations of the rank 2.

4.2.1.B Another equivalent form of this R-transformation can be written as follows,

$$\mu = -4\lambda^2(\lambda-1)(\lambda-t) \qquad \text{and} \qquad \mu - 1 = -4(\lambda - \frac{1}{\sqrt{2}})^2(\lambda + \frac{1}{\sqrt{2}})^2,$$

where $t = -1$. One can define RS-transformations by making the following choice of η-parameters: η_0 and $\eta_\infty \in \mathbb{C}$ are arbitrary and $\eta_1 = \frac{1}{2}$. This RS-transformation maps Eq. (1.1) into Eq. (1.2) with the following parameters:

$$\theta_0 = 2\eta_0, \quad \theta_1 = \theta_t = \eta_0, \quad \theta_\infty = 4\eta_\infty, \quad \text{and} \quad t = -1.$$

It is also a combination of two RS-transformations of the rank 2.

4.2.2 $RS_4^2(3+1|4|2+1+1)$

4.2.2.A

$$\mu = \frac{\rho\lambda(\lambda-1)^3}{(\lambda-t)(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-c)^4}{(\lambda-t)(\lambda-b)^2},$$

where $\rho = -\frac{2}{9} \pm \frac{i}{9\sqrt{2}}$ and

$$t = -\frac{63}{8}\rho - \frac{5}{4}, \qquad b = \frac{1}{4} + \frac{3}{4}\rho, \qquad c = -\frac{5}{4} - \frac{9}{2}\rho.$$

By making the following choice of the η-parameters: $\eta_1 = 1/4$, $\eta_\infty = 1/2$, and arbitrary $\eta_0 \in \mathbb{C}$, one defines RS-transformation which removes apparent singularities, b and c. Corresponding θ-parameters of the resulting Eq. (1.2) read:

$$\theta_0 = \eta_0, \quad \theta_1 = 3\eta_0, \quad \theta_t = \frac{1}{2}, \quad \theta_\infty = \frac{1}{2}, \quad \text{and} \quad t = \frac{1}{2} \mp \frac{7i\sqrt{2}}{16}.$$

4.2.2.B It is convenient to consider also another form of the R-transformation,

$$\mu = \frac{\rho\lambda(\lambda-a)^3}{(\lambda-t)(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-1)^4}{(\lambda-t)(\lambda-b)^2},$$

where $\rho = \frac{1}{12} \pm \frac{i}{12\sqrt{2}}$ and

$$t = 30\rho - 2, \qquad a = 32\rho - 4, \qquad b = \frac{4}{3}\rho - \frac{1}{3}.$$

Choosing the η-parameters: $\eta_0 = 1/3$, $\eta_\infty = 1/2$, and $\eta_1 \in \mathbb{C}$ is arbitrary, one defines RS-transformation which removes apparent singularities, a and b. Corresponding θ-parameters of the resulting Eq. (1.2) read:

$$\theta_0 = \frac{1}{3}, \quad \theta_1 = 4\eta_1, \quad \theta_t = \frac{1}{2}, \quad \theta_\infty = \frac{1}{2}, \quad \text{and} \quad t = \frac{1}{2} \pm \frac{5i\sqrt{2}}{4}.$$

4.2.2.C We consider here one more equivalent form of the same R-transformation,

$$\mu = \frac{\rho\lambda(\lambda - a)^3}{(\lambda - t)(\lambda - 1)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda - c)^4}{(\lambda - t)(\lambda - 1)^2},$$

where $c = -4 \pm i\sqrt{2}$ and

$$t = \frac{45}{2} + \frac{11}{2}c, \qquad \rho = \frac{c - 1}{216}, \quad a = 4c + 24.$$

Putting the η-parameters: $\eta_0 = 1/3$ or $2/3$, $\eta_1 = 1/4$ or $1/2$, and $\eta_\infty \in \mathbb{C}$ is arbitrary, one defines RS-transformation which removes apparent singularities, a and c. There are two non-equivalent RS-transformations which define two different Eqs. (1.2) with the following θ-parameters:

$$\theta_0 = \frac{1}{3} \text{ or } \frac{2}{3}, \quad \theta_1 = 2\eta_\infty, \quad \theta_t = \eta_\infty, \quad \theta_\infty = \eta_\infty, \quad \text{and} \quad t = \frac{1}{2} \pm \frac{11i\sqrt{2}}{2}.$$

4.2.3 $RS_4^2(3+1|3+1|3+1)$

$$\mu = -\frac{\lambda(\lambda - 1)^3}{4(\lambda - 1/4)^3} \qquad \text{and} \qquad \mu - 1 = -\frac{(\lambda - t)(\lambda + 1/2)^3}{4(\lambda - 1/4)^3},$$

where $t = 1/2$. RS-Transformation is defined by the following choice of the η-parameters: arbitrary $\eta_0 \in \mathbb{C}$, $\eta_1 = 1/3$, and $\eta_\infty = 1/3$ or $2/3$.

There are two non-equivalent RS-transformations which define Eq. (1.2) with the following θ-parameters:

$$\theta_0 = \eta_0, \quad \theta_1 = 3\eta_0, \quad \theta_t = \frac{1}{3}, \quad \theta_\infty = \frac{1}{3} \text{ or } \frac{2}{3}, \quad \text{and} \quad t = \frac{1}{2}.$$

4.2.4 $RS_4^2(3+1|3+1|2+2)$

4.2.4.A

$$\mu = \frac{\rho\lambda(\lambda - a)^3}{(\lambda - t)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda - 1)(\lambda - c)^3}{(\lambda - t)^2},$$

where

$$\rho = \pm\frac{3}{2}\sqrt{3}, \quad t = \frac{1}{2} - \frac{5}{27}\rho, \quad a = \frac{2}{3} - \frac{4}{27}\rho, \quad c = \frac{1}{3} - \frac{4}{27}\rho.$$

RS-Transformation is defined by the following choice of the η-parameters: $\eta_0 = 1/3$ or $2/3$, $\eta_1 = 1/3$, and $\eta_\infty \in \mathbb{C}$ is arbitrary. There are two non-equivalent RS-transformations which define Eq. (1.2) with the following θ-parameters:

$$\theta_0 = \frac{1}{3} \text{ or } \frac{2}{3}, \quad \theta_1 = \frac{1}{3}, \quad \theta_t = 2\eta_\infty, \quad \theta_\infty = 2\eta_\infty \quad \text{and} \quad t = \frac{1}{2} \mp \frac{5i\sqrt{3}}{18}.$$

4.2.4.B Another form of this R-transformation reads,

$$\mu = \frac{\lambda(\lambda-1)^3}{(\lambda-b_+)^2(\lambda-b_-)^2} \qquad \text{and} \qquad \mu - 1 = -8\frac{(\lambda-t)^3}{(\lambda-b_+)^2(\lambda-b_-)^2},$$

where $t = -1/8$ and $b_\pm = -5/2 \pm 3\sqrt{3}/2$. Corresponding RS-transformation is defined by taking arbitrary ($\in \mathbb{C}$) parameters η_0 and η_1, and putting $\eta_\infty = 1/2$. The θ-parameters of the resulting Eq. (1.2) are as follows:

$$\theta_0 = \eta_0, \quad \theta_1 = 3\eta_0, \quad \theta_t = 3\eta_1, \quad \theta_\infty = \eta_1, \quad \text{and} \quad t = -\frac{1}{8}.$$

4.2.4.C Consider here one more form of the same R-transformation,

$$\mu = \frac{\rho\lambda(\lambda-1)^3}{(\lambda-b)^2} \qquad \text{and} \qquad \mu - 1 = \frac{\rho(\lambda-t)(\lambda-c)^3}{(\lambda-b)^2},$$

where $c = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$,

$$t = 3 - 3c, \quad b = \frac{1}{4} + c, \quad \rho = -\frac{4}{9} - \frac{32}{9}c.$$

RS-Transformation is defined by taking arbitrary $\eta_0 \in \mathbb{C}$, $\eta_1 = 1/3$, and $\eta_\infty = 1/2$. Corresponding θ-parameters of Eq. (1.2) read:

$$\theta_0 = \eta_0, \quad \theta_1 = 3\eta_0, \quad \theta_t = 1/3, \quad \theta_\infty = 1, \quad \text{and} \quad t = \frac{9}{4} \mp \frac{3\sqrt{3}}{4}.$$

Note that in this case singularity of Eq. (1.2) at $\lambda = \infty$ is apparent.

4.2.5 $RS_4^2(1+1+1+1|4|4)$

R-Transformation reads:

$$\mu = \frac{\rho\lambda(\lambda-1)(\lambda-t)}{(\lambda-b)^4} \qquad \text{and} \qquad \mu - 1 = -\frac{(\lambda-c)^4}{(\lambda-b)^4}.$$

As a result of fractional linear transformation of the complex λ - plane interchanging 0, 1, and ∞, there are three sets of possible values for the parameters:

1. $c = \pm\imath, \quad b = \mp\imath, \quad \rho = \pm 8\imath, \quad t = -1;$
2. $c = 1 \pm \imath, \quad b = 1 \mp \imath, \quad \rho = \pm 8\imath, \quad t = 2;$
3. $c = \frac{1\pm\imath}{2}, \quad b = \frac{1\mp\imath}{2}, \quad \rho = \pm 4\imath, \quad t = \frac{1}{2}.$

There are two (non-equivalent) RS-transformations:

4.2.5.A The first RS-transformation is defined by taking arbitrary $\eta_0 \in \mathrm{C}$ and putting $\eta_1 = \eta_\infty = 1/4$. Resulting Eq. (1.2) has the following parameters:

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_0, \quad \theta_\infty = \eta_0, \quad \text{and} \quad t = -1,\ 2,\ \text{or } 1/2.$$

4.2.5.B The second RS-transformation is defined by taking arbitrary $\eta_0 \in \mathrm{C}$ and putting $\eta_1 = 1/4$ and $\eta_\infty = 1/2$. The parameters of the resulting Eq. (1.2) are as follows:

$$\theta_0 = \eta_0, \quad \theta_1 = \eta_0, \quad \theta_t = \eta_0, \quad \theta_\infty = \eta_0 + 1, \quad \text{and} \quad t = -1,\ 2,\ \text{or } 1/2.$$

4.2.6 $RS_4^2(2+2|2+2|2+2)$

R-transformation reads:

$$\mu = -\frac{\lambda^2(\lambda-1)^2}{(\lambda-1/2)^2} \qquad \text{and} \qquad \mu - 1 = -\frac{(\lambda-1/2-i/2)^2(\lambda-1/2+i/2)^2}{(\lambda-1/2)^2}.$$

One can define two (non-equivalent) RS-transformations:
4.2.6.A The first RS-transformation is defined by taking arbitrary ($\in$ C) η_0 and η_∞ and putting $\eta_1 = 1/2$. Resulting Eq. (1.2) has the following parameters:

$$\theta_0 = 2\eta_0, \quad \theta_1 = 2\eta_0, \quad \theta_t = 2\eta_\infty, \quad \theta_\infty = 2\eta_\infty, \quad \text{and} \quad t = 1/2.$$

4.2.6.B The second RS-transformation is defined by taking arbitrary $\eta_0 \in$ C and putting $\eta_1 = \eta_\infty = 1/2$. The parameters of the resulting Eq. (1.2) are as follows:

$$\theta_0 = 2\eta_0, \quad \theta_1 = 2\eta_0, \quad \theta_t = 1, \quad \theta_\infty = 1, \quad \text{and} \quad t = 1/2 \pm i/2.$$

In this case singularities of Eq. (1.2) at $\lambda = t$ and $\lambda = \infty$ are apparent.

Acknowledgment F. V. A. is grateful to Lev Kapitanski and Andrew Bennett for encouragement, his work was supported by NSF grants #436–2978 and CMS–9813182. A. V. K. was supported by Alexander von Humboldt-Stiftung and hosted by Universität GH Paderborn. Some results of this work were presented at Conference on Differential Equations in the Complex Domain dedicated to the memory of Professor Raymond Gérard in Strasbourg, February 20–23, 2001. We are grateful to the scientific and organizing committees for the invitation and financial support.

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